Stresses in Towed Cables during Re-entry

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Towed cables behind re-entry vehicles have been found to break progressively from the trailing end during early re-entry. An analysis of expected vibrational stresses due to periodic perturbations has been made, and this analysis has been used to estimate the variation of residual cable length with range through the atmosphere. Predictions of length variation are compared to measured values in the Naval Ordnance Laboratory 1000-ft Hypersonic Range and compare reasonably. The relative influence on cable stress and residual length are given for the various parameters of the cable and the towing body, and various methods are suggested for maximizing the length of cable to a prescribed altitude.

Nomenclature

c = cable radius
C_l = profile drag per unit length of cable
$C_{m\alpha}$ = slope of towing body moment coefficient
D = cable diameter
D_c = towing body diameter
e = 0.8 for turbulent flow and 0.5 for laminar flow
E = cable modulus of elasticity
g = acceleration of gravity
I_c = towing body transverse moment of inertia
I = cable area moment of inertia
K = constant defined by Eq. (2)
K_1 = $C_{m\alpha}$ for towing body without cable
K_2 = slope of $C_{m\alpha}$ vs L curve
K_3 = K/D
L = cable length
M = bending moment on cable
n = displacement of cable normal to s
q = dynamic pressure
r = ratio of wire diameter to cable diameter
s = length along cable
S = Laplace transform parameter
T = cable tension
t = time
u = tow body velocity
x = distance along line of flight
y = transverse deflection normal to s
Z_L = $K_2S^{0.5}/K_1$
Z_T = $K_2S^{0.8}/K_1$
α = angle of attack
β_n = root of $J_0(z)=0$
ν = kinematic viscosity
ρ_a = atmospheric density
ρ_w = cable weight per unit length
θ = angle between cable element and line of flight
σ = cable wire tensile stress
ω = forced oscillation frequency

Introduction

FOR various reasons, long cables have been towed behind re-entry vehicles in the past. Radar observations have indicated that the cables have broken off by the time the vehicle has reached an altitude of about 200,000 ft. Similarly, ground tests have indicated that trailing wires undergo violent whipping and tend to progressively break off toward the towing body. To overcome this tendency, various design changes have been made in the method of attachment, in vehicle spin rate, cable length, material properties, etc., without notable success. It is for this reason that a theoretical investigation has been made in the manner of build up of cable vibrations and in the resulting induced stresses. From the relationships derived, the relative importance of the various mechanical and material design properties will become evident.

Analysis

To obtain a feel for the response of a cable to exciting perturbations, it is perhaps best to proceed as was done at the start of this study, and examine the case of a freely hanging cable, as was originally done by Bernoulli. The tension in the cable is due to its own weight and this tension varies linearly from zero at the bottom free end to the full weight of the cable at the point of suspension. This case is different than the ordinary one of the stretched string, at all points of which there is a constant tension, and hence, a constant speed with which transverse waves propagate. In the case of the hanging cable, the propagational velocity varies linearly from a maximum at the point of suspension to zero at the lower free end. Because of the fact that the wave velocity reduces to zero at this latter end, theoretically at least, the waves never arrive there, and hence, one does not get reflection there. Instead, the vibrational energy tends to store up and one can expect violent excursions at the lower end of the cable.

The equation of motion for a cable suspended at one end is:

$$\partial/\partial x(T\partial y/\partial x) = (\rho_w/g)\ (\partial^2 y/\partial t^2) \qquad (1)$$

which is merely a statement of Newton's Law. Here x is measured along the cable length; y is the transverse displacement; T is the tension; ρ_w is the linear weight density; and g is the acceleration of gravity. Small values of $\partial y/\partial x$ are here assumed for simplicity. If the origin is taken at the bottom of the cable, at any point x, then $T=\rho_w x$. When Eq. (1) is solved by the method of separation of variables, utilizing the boundary condition $y(t,L)=0$, where L is the cable length, one finds that

$$y=\sum J_0[\beta_n(x/L)^{1/2}]\{A_n\cos[(\beta_n/2)\ (g/L)^{1/2}t] + B_n\sin[(\beta_n/2)\ (g/L)^{1/2}t]\}$$

where J_0 represents the zero-order Bessel function and β_n are the various roots of $J_0(z)=0$. Constants A_n and B_n are deter-

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mined from initial conditions at time zero. We thus see that there are an infinite number of natural frequencies

$$\omega_n = (\beta_n/2)\ (g/L)^{1/2}$$

In the case of a cable trailing behind a re-entry vehicle which is spin-stabilized, there is usually enough time for the friction at the attachment point to cause cable rotation (even if a so-called frictionless bearing is used) and hence, the cable will tend to straighten due to centrifugal force and rotate in a plane perpendicular to the vehicle spin axis. Vehicle precession will serve as one source of periodic perturbation to the cable deployment. Vehicle pitching determined by the slope of the moment coefficient is another source. The moment coefficient and perturbation frequency are in turn increased by the cable drag. Upon re-entry, the aerodynamic force normal to the cable will tend to orient the cable closer to the vehicle spin axis. The shape of the cable at an arbitrarily selected time, $t=0$, will be a function of the previous perturbation history, internal stresses due to manufacturing and storage aboard the re-entry vehicle and any previous heating prior to $t=0$. To simplify the problem, a likely shape is assumed such that the profile drag per unit length of cable is constant. Normal forces cause little tension in the cable, and since transverse vibrations are due to tension resulting from profile drag, the latter force is the only one with which we are concerned. We have thus set C_l as the tensile force per unit length of cable given by:

$$C_l = \frac{0.323\rho_a u^{3/2} \nu^{1/2} (\pi D)\cos^{3/2}\theta}{g s^{1/2}} = \frac{K\cos^{3/2}\theta}{s^{1/2}} \tag{2a}$$

for laminar flow, and

$$C_l = 0.0296\ \frac{\rho_a}{g}\ \nu^{0.2} u^{1/8} (\pi D)\ \frac{\cos^{1.2}\theta}{s^{0.2}} = \frac{K\cos^{1.2}\theta}{s^{0.2}} \tag{2b}$$

for turbulent flow where ρ_a is the air weight density at $t=0$, u = vehicle velocity, ν = kinematic viscosity, πD = circumference of cable of diameter D, s = length of cable measured from attachement point on the vehicle (Note the origin is at the other end than that of our first example), and θ = angle between a length of cable ds and the vehicle spin axis.

The shape of the cable is thus one which gives constant $\cos^{3/2}\theta/s^{1/2}$ for laminar flow and constant $\cos^{1.2}\theta/s^{0.2}$ for turbulent flow. This shape is very much as one would expect as the result of small, normal forces at the time of re-entry.

Because we have relocated the origin to the attachment point, T in Eq. (1) now becomes $C_l(L-s)$ at any point s measured along the cable. Instead of y, which was the transverse displacement in the case of the hanging cable, we now use n as the displacement normal to the aforementioned shape. Thus, Eq. (1) now becomes:

$$\partial/\partial s[C_l(L-s)\partial n/\partial s] = (\rho_w/g)\partial^2 n/\partial t^2 \tag{3a}$$

We will assume a periodic perturbation of the attachment point given by:

$$n(0,t) = n_0 \sin \omega t \tag{3b}$$

This perturbation results from a periodic pitch motion about an effective center of gravity of the towing vehicle.

The system of Eqs. (3a) and (3b) is solved by Laplace transforms giving

$$y(s,t) = n_0\omega\Bigg[\frac{[J_0(2\omega[(\rho_w/gC_l)(L-s)]^{1/2})]}{[J_0(2\omega(\rho_w L/gC_l)]^{1/2})]}(\sin\omega t)/\omega + \sum_{n=1}^{\infty}\frac{J_0(\beta_n[(L-S)/L]^{1/2})\sin[\beta_n t/(2[\rho_\omega L/gC_l]^{1/2})]}{[\omega^2-(\beta_n{}^2 gC_l)/4\rho_\omega L](\rho_w L/gC_l)^{1/2}J_l(\beta_n)}\Bigg] \tag{4}$$

which applies when ω does not equal to one of the natural frequencies; i.e., when $\omega \neq [\beta_m/2(\rho_w L/gC_l)^{1/2}]$. When, however, $\omega = [\beta_m/2(\rho_w L/gC_l)^{1/2}]$, the solution becomes

$$\begin{aligned} y(s,t) = {} & \frac{n_0}{\beta_m J_l(\beta_m)}\{-\omega t J_0(\beta_m[(L-S)/L]^{1/2})\cos\omega t \\ & + [\beta_m[(L-s)/L]^{1/2}\times J_l(\beta_m[(L-s)/L]^{1/2}) \\ & + \tfrac{1}{2}J_0(\beta_m[(L-s)/L]^{1/2})]\sin\omega t\} \\ & + n_0\omega\times\sum_{n=1,n\neq m}\frac{J_0(\beta_n[(L-s)/L]^{1/2})\sin[\beta_n t/(2[\rho_\omega L/gC_l]^{1/2})]}{[\omega^2-(\beta_n{}^2 gC_l)/4\rho_w L](\rho_w L/gC_l)^{1/2}J_l(\beta_n)} \end{aligned} \tag{5}$$

The first term in the bracket indicates the increase of amplitude with time and is the effect of resonance. It is interesting to observe that for large values of ωt and β_m, this is the important term in Eq. (5).

The next problem to be solved is the source of stress buildup. Transverse accelerations are merely the result of the cable tension which is of very small magnitude. Axial accelerations of the cable due to the transverse displacements, serve to increase the cable tension. However, the cable cannot support axial expansion taking place during half of each vibratory cycle. The axial loading is thus greatly diminished. The curvature in the cable due to the transverse vibrations leads to bending stresses. Although Eq. (1) assumes no cable stiffness exists, we can still use the curvature to calculate the cable stresses. Thus, if we approximate the curvature as $\partial^2 y/\partial s^2$, we have from the beam equation $\partial^2 y/\partial s^2 = M/EI$, where M is the bending moment, E is the modulus of elasticity, and I is the moment of inertia of the cross-sectional area. Since the stress, σ, due to bending is $\sigma = Mc/I$ where c is the distance to the outermost fiber of the cable from the center of area, then $\partial^2 y/\partial s^2 = \sigma/Ec$. Thus, using only the first term in the bracket of Eq. (5) (which is later justified in the example we have taken, since β_m is large), we obtain:

$$\sigma \approx \frac{Ecn_0\omega t(\cos\omega t)\beta_m^2 J_2(\beta_m[(L-s)/L]^{1/2})}{4J_l(\beta_m)L(L-s)} \tag{6}$$

The maximum stress occurs when $s=L$ and is given by

$$\sigma_{\max_{s=L}} \approx \frac{Ecn_0\omega t\beta_m{}^4}{32LJ_l(\beta_m)} \tag{6a}$$

The maximum stress when s is less than L is obtained as follows. In Eq. (6) $J_0(\beta_m)=0$ and hence, $J_l(\beta_m)$ is close to a maximum. Also, since β_m is large $J_l(\beta_m) \to (2/\pi\beta_m)^{1/2}$. Then σ in Eq. (6) is a maximum for $s \neq L$, when $J_2[\beta_m(L-S)/L]^{1/2}$ is a maximum. When $\beta_m[(L-S)/L]^{1/2}$ is large

$$[J_2(\beta_m[(L-S)/L))]_{\max} \to [2/(\pi\beta_m[(L-S)/L]^{1/2})]^{1/2}$$

Therefore, for $L \neq S$, maximum stress is

$$\sigma_{\substack{\max\\ s\neq L}} \approx \frac{Ecn_0\omega t\beta_m^2}{4L^{3/4}(L-S)^{5/4}} = \frac{Ecn_0\omega^3 t\rho_w}{gC_l(1-S/L)^{5/4}L} \tag{6b}$$

where we assume the cable is oriented along the line of flight.

We next calculate the variation of cable length with time or distance traveled by the towing body. We assume that the cable shortens due to stresses above the ultimate at the far end and that the whipping is due to the pitching motion of the vehicle. To simplify matters, we let $\theta_0 = 0$ at the cable end.

It is assumed that the slope of the moment coefficient for the vehicle cable assembly varies as cable drag. Thus,

$$C_{m\alpha} = K_1 + K_2 s^e \tag{7}$$

where K_1 is the $C_{m\alpha}$ of the vehicle without the cable, K_2 is a proportionality constant, and e is 0.5 for laminar flow and 0.8 for turbulent flow. The natural pitch frequency is given by

$$\omega = \left[\frac{C_{m\alpha} g \pi D_c^3 g}{4 I_c} \right]^{1/2} = \left[\frac{(K_1 + K_2 s^e) g \pi D_c^3 g}{4 I_c} \right]^{1/2} \tag{8}$$

where g is the dynamic pressure, D_c is the cone diameter, and the FPS system is used.

A value of ΔL is then calculated over which β_m applies. Thus,

$$\beta_m = 2\omega (\rho_w L / g C_1)^{1/2}$$
$$= 2([K_1 + K_2 s^e) q \pi D_c^3 g \rho_w S^{2-e}] / 4 I_c K)^{1/2}$$

Successive values of β_m vary by π since β_m is large. Thus,

$$\Delta\beta_n = \pi = 1/2 \left[\frac{g \pi D_c^3 g \rho_w}{I_c K (K_1 S^{2-e} + K_2 S^2} \right]^{1/2}$$
$$\times \; [(2-e) K_1 S^{1-e} + 2 K_2 S] \Delta L \tag{9}$$

The time for a change in length ΔL is obtained from Eq. (6b) where we use the ultimate tensile strength σ_{ult} and let $L - s = \Delta L$. Thus,

$$\Delta t = \frac{\sigma_{\mathrm{ult}} g C_1 (\Delta L)^{5/4}}{(ED/2) n_0 \omega^3 \rho_w S^{1/4} r}$$

where ΔL and ω have the values given in Eqs. (8) and (9) and r is the ratio of a wire diameter to that of the cable.

If we divide the length s into n lengths each ΔL long, $n = s/\Delta L$, and

$$\mathrm{d}n = \left[\frac{1}{\Delta L} - \frac{1}{(\Delta L)^2} \frac{\mathrm{d}(\Delta L)}{\mathrm{d}s} \right] \mathrm{d}s$$

The total time of travel becomes $t = \int \Delta t \, \mathrm{d}n$ and the distance traveled by the cable R_T and R_L for the turbulent and laminar cases, respectively, becomes

$$R_T = uT = \frac{1.0505 \; \sigma_{\mathrm{u/t}} I_c^{1.625} \nu^{.225} D^{.125} I_T}{\rho_a^{.5} U^{.225} n_0 E D_c^{4.875} \rho_w^{1.125} K_2^{.3125} K_1^{.8125} r} \tag{10a}$$

where

$$I_T = \int_{z_1}^{z_2} \frac{(z+.2935)\;(z+1.2265)}{z^{.1875} (1+z)^{2.375} (.6+z)^{1.25}} \mathrm{d}z$$

$$R_L = \frac{24.628 \sigma \nu^{.5625} I_c^{1.625} D^{.125} I_L}{\rho_a^{.5} U^{.5625} D_c^{4.875} \rho_w^{1.125} E n_0 K_1 K_2^{.625} r} \tag{10b}$$

where

$$I_L = \int_{z_1}^{z_2} \frac{(z+.5)\;(z+1.125)}{z^{.375} (z+1)^{2.375} (z+.75)^{1.25}} \mathrm{d}z$$

and

$$z = K_2 s^e / K_1$$

The limits of I_L and I_R apply to the two values of S_1 and S_2 from which and to which the cable shortening takes place.

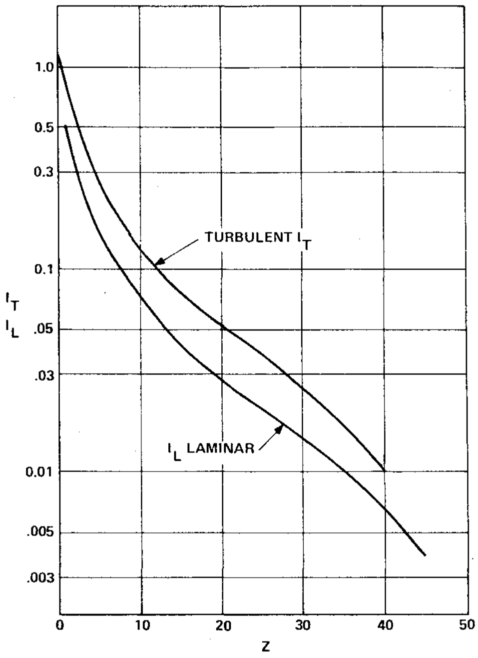

Fig. 1 Values of I_T and I_L.

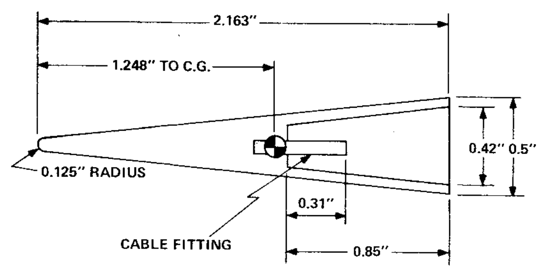

Fig. 2 Tow vehicle.

Thus $S_1 > S_2$. I_L and I_R are plotted in Fig. 1. To obtain the value, select the two values of Z which are the limits in Eq. (10) and subtract the corresponding two values of I as given in Fig. 1 from one another.

The following example will illustrate the use of Eq. (10a); and will demonstrate the agreement between calculated and experimental values. NOL has conducted a towing experiment on their 1000 ft Hyperballistics Range. The cable in the experiment continuously shortened as the towed distance increased. A 1/2-in-diam cone, trailing a 4-ft-long, .032″-diam, 7×7 steel cable (7 strands and 7 wires per strand) was launched at a velocity of 15,600 fps in an atmosphere of 24.7-torr pressure and 67°F for which $\rho_a = 0.002448$ lb/ft^3 and $\nu = 0.006287$ f^{2}ps. The mean square angle of attack was $\bar{\alpha}^2 = 20°^2$ for which $\bar{\alpha} = 4.5°$ is assumed. The Reynold's number based on cable length is approximately 5×10^6 and hence, turbulent flow prevailed. The slope of the moment coefficient was determined as $|C_{m\alpha}| = 0.027$/deg when $9'' < L < 12''$ and $|C_{m\alpha}| = 0.009$/deg when $L = o$, from which a $K_1 = 0.51566$ and $K_2 = 1.145$/rad.

Figure 2 shows the cone configuration. Its moment of inertia about a transverse axis is $I_c = 1.69$ gram in.^{2}. The distance from the end of the cable fitting to the center of gravity was 0.354 in. so that $n_0 = 0.354 \sin(4.5°)/12 = 0.002315$ ft. A value for the cable modulus of elasticity of $E = 3 \times 10^7$ psi was used, which is that of the wire because of the absence of a central fiber filler. The wire diameter is $r = 0.106$ times the cable diameter. The cable linear density was assumed to be 0.001507 lb/ft.

The results of applying Eq. (10a) to this experiment are shown in Fig. 3 for various assumed values of σ_{ult}. The best fitting value corresponds to $\sigma_{\mathrm{ult}} = 150{,}000$ psi which is a rather reasonable value for a wire of (0.106) (0.032 in.) = 0.00339 in.

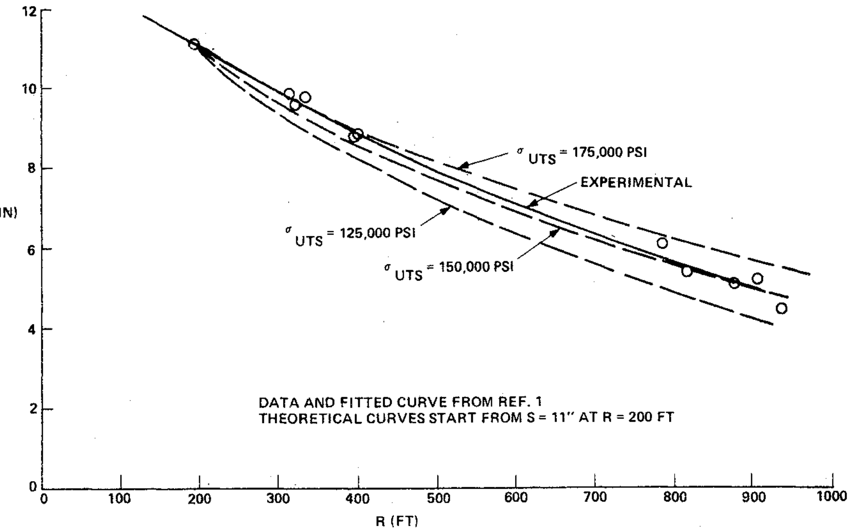

Fig. 3 Length of wire as a function of distance traveled as it breaks off during flight.

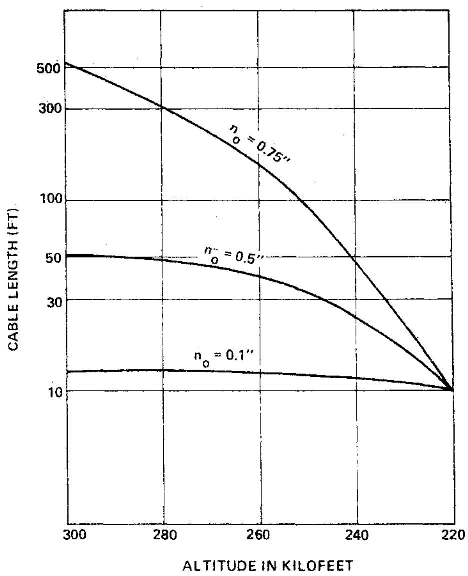

Fig. 4 Effect of attachment point amplitude on required cable length.

diameter at the elevated temperature to which the wire was subjected. No data was reported for lengths in excess of 11 in. and, hence, no comparable calculations were made.

For the case of a re-entry vehicle, it is suggested that Eqs. (10) be applied over finite altitude increments using average values of ν, ρ_a, u and n_0. R, the distance traveled by the vehicle in this altitude range, will then be available and one can solve for I_T or I_L. Knowing the value of S_1, and hence, Z_1 for the cable as it entered the altitude increment Z_2 and, hence, S_2 can be obtained from Fig. 1.

An example is given following to illustrate the application of Eq. (10) to variable altitude. At the same time, the example will emphasize the necessity of attaching the towed cable as close as possible to the towing vehicle center of gravity. It is desired that the towed cable survive to a length of 10 ft when an altitude of 220,000 ft is used. The characteristics of the vehicle are as follows: diameter = 3 ft; moment of inertia about a transverse axis through the c.g. = 2000 lb ft^2; and $C_{m\alpha}$ = 0.51566/rad. The cable specifications are: 6 strands of 7 wires each; cable diameter = 1/8 in.; cable weight = 0.0197 lb/ft; K_2 = 4.584/radian; $E_{\text{cable}} = 1.2 \times 10^7$ psi; r = 0.106, and σ_{ult} = 150,000 under heated condition. Flight conditions are 24,000 fps at a re-entry angle of 20°. Three values of n_0, the vibrational amplitude at the attachment point, are assumed: 0.75 in.; 0.5 in., and 0.1 in. to illustrate the effect of the distance of the attachment point from the vehicle c.g.

Figure 4 shows the cable length at various altitudes as it shortens to the required 10 ft at 220,000 ft. In calculating the data, increments of 20,000 ft were used over which ρ_a and ν were averaged. The range traveled in each increment was 20,000/sin 20°. The starting point was at 220,000 and the problem was worked backward to the higher altitudes. Laminar flow was assumed so that Eq. (10b) was used. Because the values of Z in Eq. (10b) were large, ΔI_L was approximated as $\Delta I_L \approx 1/Z_2 - 1/Z_1$, instead of referring to Fig. 1.

Figure 4 shows the tremendous penalty involved in allowing the attachment point to oscillate. When n_0 = 0.1 in., a 13 ft cable will suffice to provide a 10 ft length at 220,000 ft. If n_0 = 0.5, a cable close to 60 ft long appears to be needed. If n_0 is much larger, the design becomes impractical. Equation (10) will now be examined to determine the critical vehicle and cable parameters which influence the length survival of the cable.

Since cable length is predetermined by its need, we assume that it is not a parameter at the disposal of the designer. The dependence of R on the cable parameters is given by

$$R \sim \sigma_{\text{ult}} D^{1/8} / (E\ r\ \rho_w^{\ 9/8})$$

Thus, if one wishes to maximize R, he will select a cable of greater ultimate strength consisting of wires as small as possible in a cable diameter as large as possible (although cable diameter is of little importance). The cable modulus should be as low as possible. The parameter K_2 which is a measurement of increase $C_{m\alpha}$ with profile drag affects R to a slight extent only.

The vehicle parameters product the following effect on R : $R \sim I_c^{\ 13/18} / n_0 D_c^{\ 39/8}$ together with a dependence on K_1 which is the $C_{m\alpha}$ with no cable attached. Since I_c and D_c are determined due to other considerations, the chief parameter in our control is n_0. Thus, to minimize n_0, attachment as close as possible to the vehicle center of gravity seems to be the most advisable design aim.

Finally, R will vary as $\nu^{.225}/\rho_a^{\ .5}$ for turbulent flow and as $\nu^{.5625}/\rho_a^{\ .5}$ for laminar flow, and will thus decrease with decreasing altitude. This would indicate that an artifical pull on the cable, as by a drag brake, will increase the tendency of the cable end to break off faster.

References

[1] Sheetz, N. W. and Krumins, M. V., "An Investigation of the Effect of Flexible Trailing Wire on a Hypervelocity Cone," NOLTR 70-49, March 1970, Naval Ordnance Lab., White Oak, Silver Spring, Md.